Models of imbalanced MHD turbulence

Grigol Gogoberidze[*,†], Stefaan Poedts[*] and Mariam Akhalkatsi[†]

[*]*Centre for Plasma Astrophysics, K.U.Leuven, Celestijnenlaan 200B, 3001 Leuven, Belgium*
[†]*Institute of Theoretical Physics, Ilia State University, 3/5 Cholokashvili str., 0162 Tbilisi, Georgia*

Abstract. The relation between the energy imbalances and the dissipation rate imbalances is derived for the model of strong MHD turbulence which implies incoherent straining imposed by subdominant waves on a dominant wave packet and a pinning of the spectra of counter propagating Alfvén waves at the dissipation scale. The comparison of the obtained result to the results of recent numerical simulations shows that the fitting is poor both for the weakly and strongly imbalanced cases.

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INTRODUCTION

Magnetohydrodynamic (MHD) turbulence is one of the key ingredients of many astrophysical objects, including the solar wind and the solar corona [1, 2]. One of the first observations of MHD turbulence in the solar wind showed that the turbulence is strongly imbalanced, in the sense that it is dominated by the waves propagating outward from the sun [3]. In other words, this means that the turbulence in the solar wind has nonzero cross helicity, which is one of the conserved quantities of ideal MHD.

It is long known [4, 5] that cross helicity plays an important role in the dynamics of MHD turbulence. One important feature is that the conservation of energy and cross helicity implies the separate conservation of the energies of waves propagating parallel and antiparallel to the mean magnetic field. Due to this reason, positive and negative waves (under positive/negative waves we imply the waves propagating parallel/antiparallel to the mean magnetic field) can have different cascade features. Recently, several phenomenological models of strong anisotropic MHD turbulence with nonzero cross helicity (so-called imbalanced MHD turbulence) have been developed [6, 7, 8, 9, 10]. The different models yield different predictions regarding the spectral slopes of positive and negative waves as well as the relations between the various global characteristics of the turbulence, such as energy dissipation rates, total energies of plus and minus waves, etc.

High resolution direct numerical simulations (DNS) are of great importance to check the various predictions of the different models of turbulence. However, in the case of MHD turbulence, the determination of the spectral slopes by DNS has significant limitations. Firstly, the influence of nonlocal interactions seems to be much more important for MHD turbulence then for hydrodynamic turbulence [11, 12]. Due to this reason, in the currently available DNS, where the inertial interval has a very limited extent, a reliable measurement of the spectral slopes of the MHD turbulence is much more difficult then in the case of hydrodynamic turbulence. Secondly, imbalanced MHD turbulence needs much more time for relaxation to the stationary state, which makes it impossible to study the energy spectra of very strongly imbalanced MHD turbulence using present day computing power [9, 13].

Recently however, Beresnyak and Lazarian [14] have noticed that the ratio of the energy injection/dissipation rates is a much more robust quantity then the spectral indices and, therefore, it can be used to differentiate among various models of strong imbalanced MHD turbulence. These authors studied strong imbalanced MHD turbulence for different energy injection rates and determined how the ratio of the energy injection rates (energy injection imbalance) depends on the energy ratio of the counter propagating Alfvén waves (i.e., the energy imbalance). A comparison of the results with predictions of the models developed in Refs. [6] and [9] has shown that for small imbalances agreement the prediction with the LGS model is good, whereas for stronger imbalances, for a given ratio of energy injection rates, numerical simulations show a much larger energy ratio then predicted by either of the models. In the presented paper, we derive the relation between the energy injection imbalance and the energy imbalance of the counter propagating Alfvén waves predicted by another type of models, which assumes incoherent straining imposed by subdominant waves on a dominant wave packet and the pinning effect at dissipation scales (e.g., [8]). Then, the obtained results are compared to the results observed in DNS of Beresnyak and Lazarian [14].

The paper is organized as follows. The derivation of the above mentioned relation is presented in Sec. 2. A discussion and conclusions are given in Sec. 3.

THEORETICAL TREATMENT

Let us assume that w^+ denotes the Elsasser variable of the dominant perturbations. In Ref. [6] it was shown that the cascade time for w^- perturbations at some perpendicular (with respect to the mean magnetic field) scale $k_\perp \sim 1/l$ is $\tau_l^- \sim 1/kw_l^+$, where w_l^+ denotes the typical value of w^+ perturbations with length scale l. Because the amplitudes of the negative waves are smaller compared to the amplitudes of the positive waves, a collision between positive and negative waves packets is not sufficient to cascade the energy of the positive waves to smaller scales. The key feature of the model developed in Ref. [6] is the assumption that the straining rate imposed by different negative wave packets on the positive ones is imposed coherently. If this is the case, then the spectral slope is $-5/3$ for both positive and negative waves , and the energy cascade rates are $\varepsilon^\pm \sim (w_l^\pm)^2 w_l^\mp/l$. Hence, for the amplitude ratio at the energy injection scale this model predicts

$$\frac{w_0^+}{w_0^-} \sim \frac{\varepsilon^+}{\varepsilon^-}. \tag{1}$$

The model developed in Ref. [9] is based on the 'scale dependent dynamic alignment effect'. According to this model, MHD turbulence consists of different domains with positive and negative alignment between the velocity and magnetic fields of the perturbations. For such a domain, the model predicts $w_0^+/w_0^- \sim (\varepsilon^+/\varepsilon^-)^{1/2}$. It was argued in Ref. [9], that this local relation does no necessarily hold globally. Although this is true in general, due to the fact that both the energy transfer rates and the energies are additive physical quantities, in the strongly imbalanced case, when the total volume occupied by the turbulence should be strongly dominated by positively aligned domains, one can expect that the above presented relation between the fluxes and the energies should hold approximately.

According to the models developed in Refs. [7] and [8], the straining of different negative wave packets on a positive wave packet is not coherent and, consequently, in the framework of these models the cascade rates have the same scaling

$$\varepsilon^\pm \sim (w_l^-)^2 w_l^+/l. \tag{2}$$

Due to this degeneracy, in this kind of models Kolmogorov-like arguments and some condition which determines the order of the anisotropy (like the critical balance condition of Ref. [16]) alone are not sufficient to determine the spectral slopes. As needed extra condition, the so-called 'pinning' effect was introduced in Ref. [5] when studying isotropic imbalanced turbulence. Based on the fact that, in the framework of MHD, nonlinear interactions are only possible among counter propagating Alfvén waves, it has been shown that the dissipation length scales of the positive and negative waves should be equal [5]. This leads to the conclusion that the rms amplitudes of the waves also should be equal at the dissipation scale. Chandran [8] applied the same principle to the anisotropic imbalanced MHD turbulence and derived expressions for the spectral slopes. Suppose the one dimensional perpendicular energy spectra are $E^\pm(k_\perp) \sim k_\perp^{m_\pm}$. Taking into account that $E^\pm(k_\perp)k_\perp \sim (w_l^\pm)^2$, Eq. (2) yields [7]

$$m_+ + 2m_- = 5. \tag{3}$$

According to the pinning effect

$$E^\pm(k_\perp) = E_d \left(\frac{k_\perp}{k_d}\right)^{m_\pm}, \tag{4}$$

where $2\pi/k_d$ is the dissipation length scale. Equation (4) yields

$$m_+ - m_- = \frac{2\log(w_0^+/w_0^-)}{\log(k_d/k_0)}. \tag{5}$$

Here, k_0 denotes the minimal perpendicular wave number where the power law spectrum is still valid (note, that this can be significantly different from the mean injection wave number or the wave number where the energy spectrum peaks). The $w_0^\pm$ denote the rms amplitudes at length scale $1/k_0$. Equation (5), together with Eq. (3), enables one to determine the spectral slopes predicted by the model [8]. The model developed in Ref. [7] does not imply the pinning effect explicitly, but the effects related with the difference in spectral slopes are included in function $f(\lambda)$ in their

Eq. (4). In addition, this model implies a different order of anisotropy of the dominant and subdominant wave packets. We do not consider this effect here and will comment on this subject in the next section.

Our goal is to determine the relation between w_0^+/w_0^- and $\varepsilon^+/\varepsilon^-$. This is not a trivial task and can be done in several ways. In the framework of weak turbulence theory this is usually done by an explicit evaluation of integral expressions describing the nonlinear transfer of energy [15, 17]. Another approach was used in the phenomenological advection-diffusion model developed in Ref. [8]. Due to the fact that both plus and minus fluxes has the same scaling, Chandran [8] introduced a weighting coefficient (denoted in his paper by c_1) and weighting functions ($h_k^{\pm}$) in such a way, that in the limit of weakly imbalanced turbulence they reduce to the results obtained in Ref. [15] in the framework of the weak turbulence theory. As a consequence, the relation between the injection rate imbalance and energy imbalance which can be obtained in the framework of the advection-diffusion model, would be restricted to the weakly imbalanced and weakly turbulent case. Another possibility to connect the energy fluxes with amplitudes at the injection scale is to use the expressions for the energy dissipation rates (which in the stationary case obviously coincide with the energy injection rates), which are defined as

$$\varepsilon^{\pm} = \nu \int \mathrm{d}\mathbf{k}^3 k^2 \mathscr{E}^{\pm}(\mathbf{k}) = \nu \int \mathrm{d}k_\perp k_\perp^2 E^{\pm}(k_\perp), \tag{6}$$

where $\mathscr{E}^{\pm}(\mathbf{k})$ denotes the three dimensional energy spectrum. This method was first used in Ref. [5] for the study of isotropic imbalanced MHD turbulence. It must be underlined that this is not a unique 'derivation' of the needed relation between the rms amplitudes and the dissipation rates, but rather another assumption. Indeed, for $k_0 \ll k_d$ the integrals in Eq. (6) are strongly dominated by the contribution from the upper boundary, and consequently, using these expressions for the relation w_0^+/w_0^- with $\varepsilon^+/\varepsilon^-$ implicitly implies, that at deep inertial interval, close to the dissipation scale the energy spectrum still can be satisfactorily described by Eq. (4). Substituting Eq. (4) into Eq. (6), we obtain

$$\frac{\varepsilon^+}{\varepsilon^-} = \frac{(3-m_-)[1-(k_0/k_d)^{3-m_+}]}{(3-m_+)[1-(k_0/k_d)^{3-m_-}]}. \tag{7}$$

Combining this equation with the Eqs. (3) and (5) we obtain for the relation between w_0^+/w_0^- and $\varepsilon^+/\varepsilon^-$, viz.

$$\frac{\varepsilon^+}{\varepsilon^-} = \frac{\ln(\bar{k}^4 \bar{E}_0)}{\ln(\bar{k}^4 \bar{E}_0^{-2})} \frac{1-\bar{k}^{-4/3}\bar{E}_0^{2/3}}{1-\bar{k}^{-4/3}\bar{E}_0^{-1/3}}, \tag{8}$$

where $\bar{k} = k_0/k_d$ and $\bar{E}_0 = (w_0^+/w_0^-)^2$. This equation represents the main result of the present paper.

In the weakly imbalanced case, i.e., when $(\varepsilon^+ - \varepsilon^-)/\varepsilon^- \ll 1$, Eq. (8) yields

$$\frac{(w_0^+)^2 - (w_0^-)^2}{(w_0^-)^2} \sim \frac{4\log(k_d/k_0)}{3}\left(\frac{\varepsilon^+}{\varepsilon^-} - 1\right). \tag{9}$$

Although Eq. (8) gives finite results for any energy ratio, its applicability is limited to the values $\bar{E} \le \bar{k}^2$. Indeed, if $\bar{E}_0^2/\bar{k}^4$ becomes greater then unity, then m_2 becomes smaller then 1, and nonlocal interactions start to dominate the energy cascade. For the highest possible imbalance $\bar{E}_0^2 = \bar{k}^4$ we have

$$\frac{\varepsilon^+}{\varepsilon^-} \approx 2\ln\bar{k}. \tag{10}$$

DISCUSSION AND CONCLUSIONS

The principal distinction between Eq. (8) and the predictions of other models is the strong dependence of the relation on the ratio k_d/k_0, or equivalently, on the Reynolds number. This is due to the pinning effect which implies that the dissipation scale dynamics significantly affects the dynamics in the inertial interval.

The comparison of the obtained result with the predictions of other models and the results of the DNS reported in Ref. [14] are presented in Fig. 1. The upper line corresponds to Eq. (8) for $\bar{k} = 10$ (as was mentioned above k_0 is not the injection wave number, but rather the minimal wave number where the power law spectrum holds. Due to the reason that the inertial interval in the currently available simulations have very limited extent, the chosen value

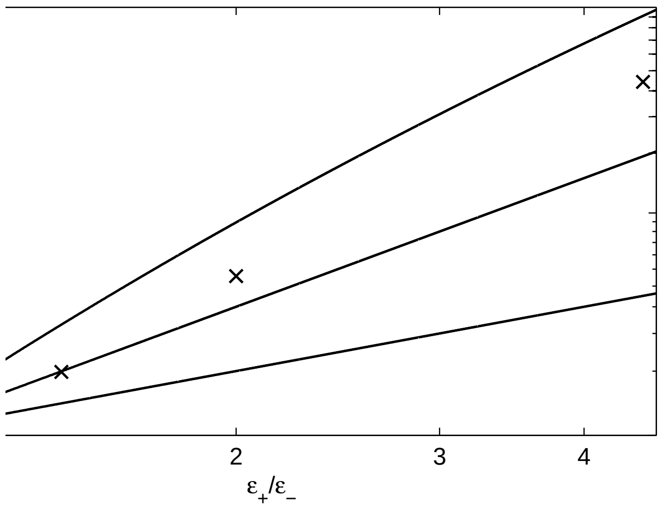

FIGURE 1. Energy imbalance vs dissipation rate imbalance. The upper line corresponds to the Eq. (8) for $\bar{k} = 10$. The middle line is the prediction of Ref. [6] and dash-dotted line corresponds to the relation $w_0^+/w_0^- \sim (\varepsilon^+/\varepsilon^-)^{1/2}$. Crosses represent the results of DNS from Ref. [14].

of k_0 seems us reasonable). The middle line is the prediction of Ref. [6] and lower line corresponds to the relation $w_0^+/w_0^- \sim (\varepsilon^+/\varepsilon^-)^{1/2}$. Crosses represent the results of the DNS from Ref. [14].

It can be seen from Fig. 1, that the fitting of Eq. (8) with numerical results is poor both in weakly and in strongly imbalanced cases. For a given injection rate imbalance, Eq. (8) predicts a higher ratio w_0^+/w_0^- then is observed in the DNS. This means that the cascade of dominant waves observed in the simulations is stronger then is predicted by the model.

In this paper we do not consider the parallel dynamics of the perturbations, implicitly assuming that it is slaved by the perpendicular dynamics and also assuming that both dominant and subdominant waves have the same order of anisotropy. The model developed in Ref. [7] implies that wave packets of positive and negative waves have a different order of anisotropy, and this feature can affect the turbulent dynamics. In principle, this can lead to some 'strengthening' of the dominant wave cascade and, consequently, make the theoretical predictions more suitable with the results of DNS. This topic requires more detailed study. Another possible explanation of the observed inconsistency (especially for the strongly imbalanced case) can be related to the fact that as mentioned above, the imbalanced MHD turbulence requires much more time for relaxation to the stationary state then the balanced turbulence. Therefore, the results obtained using the formalism of stationary turbulence may not be suitable to describe properly the results of the considered DNS.

Summarizing, we obtained a relation between the ratio of the energy injection rates and the ratio of the energies predicted by the models of strong imbalanced MHD turbulence, which assume incoherent straining imposed by subdominant waves on a dominant wave packet and the pinning effect at the dissipation scales. Due to the pinning effect, the relation (in contrary to other models of strong imbalanced MHD turbulence) strongly depends on the Reynolds number. The obtained result is in poor agreement with the results observed in recent direct numerical simulations.

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